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Journal of Geometry and Physics 42 (2002) 259–282

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Noncommutative Riemannian geometry of the alternating group $\mathcal{A}_4$

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Received 6 September 2001

Abstract

We study the noncommutative Riemannian geometry of the alternating group $\mathcal{A}_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ using the recent formulation for finite groups. We find a unique 'Levi-Civita' connection for the invariant metric, and find that it has Ricci flat but nonzero Riemann curvature. We show that it is the unique Ricci flat connection on $\mathcal{A}_4$ with the standard framing (we solve the vacuum Einstein's equation). We also propose a natural Dirac operator for the associated spin connection and solve the Dirac equation. Some of our results hold for any finite group equipped with a cyclic conjugacy class of four elements. In this case the exterior algebra $\Omega(\mathcal{A}_4)$ has dimensions 1:4:8:11:12:12:11:8:4:1 with top-form nine-dimensional. We also find the noncommutative cohomology $H^1(\mathcal{A}_4) = \mathbb{C}$.

MSC: 46L87; 58B32

Keywords: Noncommutative Riemannian geometry; Quantum groups; Finite groups; Quantum gravity

Subj. Class.: Noncommutative geometry; Quantum groups

1. Introduction

A constructive formalism of noncommutative Riemannian geometry has recently been developed in [1,2], using quantum group methods. Here a general and possibly noncommutative algebra or 'coordinate ring' is equipped with a 'quantum Riemannian manifold'

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0393-0440/02/$ – see front matter
PII: S0393-0440(01)00089-4

structure consisting of a frame bundle (with quantum group fibre) to which the differential calculus [3] or exterior algebra bundle is associated. The approach builds on the established theory of quantum principal bundles [4] and adds to this notions of ‘framing’, ‘coframing’ (or metric) and Levi-Civita type metric-compatible connections with Riemann and Ricci curvature.

This approach applies also to finite sets and finite groups, and allows one to endow them with nontrivial Riemannian manifold structures. This would not be possible within conventional differential geometry, but noncommutative differential structures are more general even when the coordinate ring is commutative, and allow a rich structure even for finite sets. Particularly, for a finite group there are natural choices for translation bi-invariant differential structure, namely labelled by the nontrivial conjugacy classes (see [3]). There is a natural frame bundle, namely with fibre another copy of the group [1], and there is a natural ‘Killing form’ inducing a metric [2], which may, however, be degenerate. So far, only the case of S_3 in [2] has been worked out in detail it is shown there that this has the natural structure of a noncommutative Einstein manifold with unique Levi-Civita connection for the Killing metric, and with Ricci curvature essentially proportional to the metric. Some other aspects of the noncommutative geometry of S_3 are in [5].

In this paper we extend the repertoire of examples with a detailed study of the alternating group $\mathcal{A}_4$ from this point of view. It turns out that this example is of key interest because, like S_3, it has a natural invariant metric with unique Levi-Civita connection, but this connection is Ricci flat. Thus, it provides the first concrete example of the solution of the vacuum Einstein equations in this theory. The model is also interesting because the group is nonabelian enough to have nontrivial curvature (the Riemann tensor does not vanish) but simple enough to be fully computable.

Some of the computations are done without reference to the actual group and apply to any group with similar ‘cyclic’ conjugacy class. We include for example $\mathrm{SL}(2, \mathbb{Z}_3)$ in this family. After some preliminaries, we start in Section 3 at this general level. We find the Woronowicz exterior algebra associated to the conjugacy class and show that it is *not* in fact quadratically generated. It is in fact a good example where we see the absolute necessity of relations in higher degree. We then find the unique form of the invariant metric, namely

$$\eta^{a,b} = \delta_{a,b} + \mu,$$

in a suitable basis, with μ a parameter. We also characterise the torsion-free and cotorsion-free regular connections. In Section 4 we specialise to $\mathcal{A}_4$ and find the associated Levi-Civita or metric-compatible connection and its Ricci flatness. We also show that it is the unique regular Ricci flat connection on $\mathcal{A}_4$ independently of metric compatibility.

In Section 5 we look at the Dirac operator appropriate to the metric, as a step towards comparison with Connes’ approach to noncommutative geometry [6]. As in [2] for S_3, it is not naturally Hermitian but is fully diagonalisable. Finally, Section 6 contains some further results including the noncommutative de Rahm cohomology and a link to the differential (but not Riemannian) geometry of S_4 in [7], as well as concluding remarks.

We note that following [1] there have been some other attempts at a Riemannian geometry on finite groups, but different from the one used here, see for instance [8,9] and references therein. While the use of a conjugacy class to define an exterior algebra, i.e. the notion of differential structure, is the same in all approaches following [3], see for instance [10], the formulation of [1,2] is the only one that features some kind of metric compatibility or 'Levi-Civita' notion for the spin connection, as well as the only one that features a nontrivial (and nonuniversal) differential structure in the fibre direction, and hence an actual noncommutative geometry of the frame bundle. Other problems solved in [2] were the formulation and computation of the Ricci tensor, see [11] for a discussion.

2. Preliminaries

In this section, we briefly recall the basic definitions of differential structures [3] and of noncommutative Riemannian geometry [2], specialised to the case of finite groups that we need in the paper. Thus, we work with the Hopf algebra $H=\mathbb{C}[G]$ of functions on a finite group G. We equip it with its standard basis $\{\delta_g\}_{g\in G}$ defined by Kronecker delta functions $\delta_g(h)=\delta_g^h, \forall g,h\in G$. Let $\mathcal{C}$ be a nontrivial conjugacy class of G, and Ω_0 the vector subspace

$$\Omega_0=\{\delta_a|a\in\mathcal{C}\}=\mathbb{C}\mathcal{C}. \tag{1}$$

Any Ad-stable set not containing the group identity can be used here, but we focus on the irreducible case of a single conjugacy class. From the Woronowicz's theory in [3], the first-order differential calculus associated to $\mathcal{C}$ is generated over $\mathbb{C}[G]$ by Ω_0 and given by

$$\mathrm{d}f=\sum_{a\in\mathcal{C}}(\partial^a f)e_a,\quad \partial^a=R_a-\mathrm{id},\ \ e_af=R_a(f)e_a,\ \ \forall a\in\mathcal{C},\ f\in H, \tag{2}$$

where the operator R_a is defined by $R_a(f)(g)=f(ga), \forall g\in G$. The Maurer–Cartan one-form $e:\Omega_0\to\Omega^1(H)$ is given by

$$e_a=e(\delta_a)=\sum_{g\in G}\delta_g\,\mathrm{d}\delta_{ga}, \tag{3}$$

and higher differential forms are obtained from Woronowicz skew-symmetrisation procedure [3], using the braiding

$$\Psi(e_a\underset{H}{\otimes}e_b)=e_{aba^{-1}}\underset{H}{\otimes}e_a. \tag{4}$$

The Maurer–Cartan equation takes the form

$$\mathrm{d}e_a=\theta\wedge e_a+e_a\wedge\theta,\quad \theta=\sum_{a\in\mathcal{C}}e_a. \tag{5}$$

The element θ obeys $\theta\wedge\theta=0$ and generates 'd' in general as graded commutator with θ. Lemma 5.3 in [2] gives the full set of relations of $\Omega^2(H)$, namely

$$\sum_{a,b\in\mathcal{C};ab=g}\lambda_a^{g,\beta}e_a\wedge e_b=0,\quad \forall g\in G,\ \ \forall\beta, \tag{6}$$

where for $g\in G$ fixed, $\{\lambda^\beta\}$ is a basis of the invariant subspace of the vector space spanned by $\mathbb{C}C\cap g\mathbb{C}C^{-1}$ under the automorphism $\sigma(a)=a^{-1}g$. There are also cubic and higher degree relations (which are in fact nontrivial in our case of $\mathcal{A}_4$) but we will not need them explicitly (most of Riemannian geometry needs only one-forms and two-forms).

Next, following [2], a framing means a basis of $\Omega^1(H)$ over $\mathbb{C}[G]$, and an action of the frame group. In our case we chose the framing to be the components $\{e_a\}$ of the Maurer–Cartan form as above and for frame group we choose G itself, acting by Ad. This is a canonical choice and its classical meaning is explained in [1]. A spin connection is then a collection $\{A_a\}_{a\in\mathcal{C}}$ of component one-forms. Its associated covariant derivative is defined on a one-form $\alpha = \alpha^a e_a$ by

$$\nabla\alpha = \mathrm{d}\alpha^a \underset{H}{\otimes} e_a - \alpha^a \sum_{b\in\mathcal{C}} A_b \underset{H}{\otimes} (e_{b^{-1}ab} - e_a) \tag{7}$$

with summation on a. The associated torsion tensor $T : \Omega^1(H) \to \Omega^2(H)$ is defined by $T\alpha = \mathrm{d}\wedge\alpha - \nabla\alpha$ and the zero-torsion condition is vanishing of

$$\bar{D}_A e_a \equiv \mathrm{d}e_a + \sum_{b\in\mathcal{C}} A_b \wedge (e_{b^{-1}ab} - e_a), \quad \forall a \in \mathcal{C}. \tag{8}$$

The spin connection here has values in the dual space Ω_0^*, which is a 'braided-Lie algebra' in a precise sense. Associated to this geometrical point of view, there is a regularity condition

$$\sum_{a,b\in\mathcal{C};ab=g} A_a \wedge A_b = 0, \quad \forall g \neq e, \ \ g \notin \mathcal{C}. \tag{9}$$

The curvature ∇^2 associated to a regular connection A is in frame bundles terms a collection of two-forms $\{F_a\}_{a\in\mathcal{C}}$ defined by

$$F_a = \mathrm{d}A_a + \sum_{c,d\in\mathcal{C},cd=a} A_c \wedge A_d - \sum_{c\in\mathcal{C}} (A_c \wedge A_a + A_a \wedge A_c), \tag{10}$$

while the Riemann curvature $\mathcal{R} : \Omega^1(H) \to \Omega^2(H)\otimes_H \Omega^1(H)$ is given by

$$\mathcal{R}\alpha = \alpha^a \sum_{b\in\mathcal{C}} F_b \underset{H}{\otimes} (e_{b^{-1}ab} - e_a). \tag{11}$$

Finally, the Ricci tensor is given by

$$\mathrm{Ricci} = \sum_{a,b,c\in\mathcal{C}} i(F_c)^{ab} e_b \underset{H}{\otimes} (e_{c^{-1}ac} - e_a), \tag{12}$$

where $i(F_c) = i(F_c)^{ab} e_a \otimes_H e_b$ and $i : \Omega^2(H) \to \Omega^1(H)\otimes_H \Omega^1(H)$ is a lifting which splits the projection of $\wedge$. A canonical choice is [2]

$$i(e_a \wedge e_b) = e_a \underset{H}{\otimes} e_b - \sum_\beta \gamma^{\beta,a} \sum_{c,d\in\mathcal{C},cd=ab} \lambda_c^\beta e_c \underset{H}{\otimes} e_d, \tag{13}$$

where $\{\gamma^\beta\}$ are the dual basis to the $\{\lambda^\beta\}$ with respect to the dot product as vectors in $\mathbb{C}\mathcal{C} \cap ab\mathcal{C}^{-1}$. Another canonical 'lift' is $i' = \mathrm{id} - \Psi$ but note that in this case $i' \circ \wedge$ is *not* a projection operator.

There are two further structures that one may impose in this situation. First of all, given a choice of framing $\{e_a\}$, a metric g is defined as a coframing $\{e^{*a}\}$, i.e. again a basis of Ω^1 but now as a right $\mathbb{C}[G]$-module and transforming under the contragradient action of G. (The corresponding metric is $g = \sum_a e^{*a} \otimes_H e_a$.) The cotorsion of a spin connection is the torsion with respect to the coframing, and is given by

$$D_A e^{*a} \equiv \mathrm{d}e^{*a} + \sum_{b\in\mathcal{C}} (e^{*bab^{-1}} - e^{*a}) \wedge A_b. \tag{14}$$

Vanishing of cotorsion has the classical meaning of a generalisation of metric compatibility of the spin connection, see [1]. So we are usually interested in regular torsion-free and cotorsion-free connections.

Finally, a 'gamma-matrix' is defined [2] as an equivariant collection of endomorphisms $\{\gamma_a\}_{a\in\mathcal{C}}$ of a vector space W on which G acts by a representation ρ_W, a 'spinor field' is a W-valued function on G and the Dirac operator on the spinor fields is

$$\not{D} = \partial^a \gamma_a - A^b_a \gamma_b \tau^a_W, \tag{15}$$

where $A_b = A^a_b e_a$ and $\tau^a_W = \rho_W(a^{-1} - e)$. There is a canonical choice where γ is built from ρ_W itself, explained in [2].

3. Cyclic Riemannian structures

In this section, we construct Riemannian geometry on groups endowed with conjugacy classes which obey a certain cyclicity condition. For the case when the differential calculus is of degree 4, we determine the entire exterior algebra and the moduli space of torsion-free connections, and for any degree $n \geq 2$, we give the general form of the invariant metric.

Definition 3.1. Let $\mathcal{C}$ be a conjugacy class with n elements, $n \geq 2$, in a group G. We say that $\mathcal{C}$ is 'cyclic' if there exists at least one t in $\mathcal{C}$ such that Ad_t is a cyclic permutation of $\mathcal{C} - \{t\}$ and the map $a \mapsto \mathrm{Ad}_a(t)$ is a permutation of $\mathcal{C}$.

For $n = 4$ we have the following characterisation of $\Omega^2(H)$.

Proposition 3.2. *For a cyclic conjugacy class $\mathcal{C} = \{t, x, y, z\}$ of order 4 in a finite group G, the bimodule $\Omega^2(H)$ of two-form is eight-dimensional over $\mathbb{C}[G]$ and is defined by the following equations*:

$$e_a \wedge e_a = 0, \qquad \sum_{a,b\in\mathcal{C};ab=g} e_a \wedge e_b = 0, \quad \forall a \in \mathcal{C}, g \in G, \tag{16}$$

where $\{e_a\}_{a\in\mathcal{C}}$ is the basis of Maurer–Cartan one-forms.

Proof. We assume the existence of an element $t \in \mathcal{C}$ as in Definition 3.1. Without loss of generality, we denote the other elements of $\mathcal{C}$ by x, y, z with Table 1 for Ad. It follows that

$$tx = zt = xz, \qquad ty = xt = yx, \qquad tz = yt = zy, \qquad xy = zx = yz. \tag{17}$$

Table 1

Ad	t	x	y	z
t	t	z	x	y
x	y	x	z	t
y	z	t	y	x
z	x	y	t	z

Using relations (17), we apply the Woronowicz antisymmetrisation procedure to obtain the following relations of the form (16) in $\Omega^2(H)$:

$$\begin{aligned}&e_a\wedge e_a=0,\quad \forall a\in\mathcal{C},\qquad e_t\wedge e_x+e_x\wedge e_z+e_z\wedge e_t=0,\\&e_x\wedge e_t+e_t\wedge e_y+e_y\wedge e_x=0,\qquad e_t\wedge e_z+e_z\wedge e_y+e_y\wedge e_t=0,\\&e_x\wedge e_y+e_y\wedge e_z+e_z\wedge e_x=0.\end{aligned}\tag{18}$$

This form (16) holds for any group since the elements $e_a\otimes e_a$ and $\sum_{ab=g}e_a\otimes e_b$ are in the kernel of $\mathrm{id}-\Psi$. However, using (6) one may see that they are the *only* relations of $\Omega^2(H)$ which is therefore of dimension 8 as stated. □

From now, we choose a basis of $\Omega^2(H)$ to be

$$\{e_t\wedge e_x, e_t\wedge e_y, e_t\wedge e_z, e_x\wedge e_t, e_y\wedge e_t, e_x\wedge e_y, e_y\wedge e_z, e_x\wedge e_z\}.\tag{19}$$

For convenience, we will sometimes use indexes 1–4 to refer to t, x, y, z, respectively.

Proposition 3.3. *In the setting of* Proposition 3.2, *the dimensions of the Woronowicz exterior algebra* $\Omega(H)$ *are* 1:4:8:11:12:12:11:8:4:1 *with top-form of degree* 9. *This algebra is not quadratic, having additional relations in degree* ≥ 6.

Proof. As above, we do not need the group itself but only the matrix for Ad restricted to the conjugacy class (i.e. Table 1). In fact we are computing the invariant part Λ of the exterior algebra, with $\Omega(H)=H\otimes\Lambda$ as a vector space. This Λ is generated over $\mathbb{C}$ by the $\{e_a\}$ with relations determined by the braiding Ψ. Namely we set to zero the kernels of the antisymmetisers A_m for $m\geq 2$. These A_m are described in [3] as a signed sum over permutations of $\{1,\dots,m\}$ with transposition replaced by Ψ. This is not very convenient for computation and we employ instead a different but equivalent definition of the A_m coming out of the theory of braided groups [12]. As recently discussed in [7], we use the braided-integers

$$\begin{aligned}[m,-\Psi]&=\mathrm{id}-\Psi_{12}+\Psi_{12}\Psi_{23}-\cdots\pm\Psi_{12}\cdots\Psi_{m-1,m}\\&=\mathrm{id}-\Psi_{12}(\mathrm{id}\otimes[m-1,-\Psi]),\end{aligned}$$

where Ψ_{12} denotes Ψ acting in the first and second places of $\Omega_0^{\otimes m}$, etc. Then

$$A_m=[m,-\Psi]!=(\mathrm{id}\otimes[2,-\Psi])(\mathrm{id}\otimes[3,-\Psi])\cdots[m,-\Psi].$$

In the braided groups approach to the exterior algebra we set to zero the kernels of all these braided factorials. It is straightforward to program these inductive definitions. We first compute the 16×16 matrices Ψ acting in the tensor product basis $e_a \otimes e_b$ and then the A_m as above, up to A_6. The dimensions of $\Omega^m(H)$ over H are then $4^m - \dim\ker A_m$ and found to be as stated. From the general form expected for the exterior algebra we assume the remaining dimensions for A_7, A_8, A_9 without explicit computation. Finally, the quadratic exterior algebra is defined by setting to zero only the kernel of $A_2 = \mathrm{id} - \Psi$ without additional relations in higher degree. In that case in degree m we set to zero the union of the null spaces $\mathrm{id} - \Psi_{12}, \dots, \mathrm{id} - \Psi_{m-1,m}$. Here we find dimensions 1:4:8:11:12:12:12:$\cdots$, i.e. fewer relations in degree ≥ 6 (it appears that the quadratic one is in fact infinite-dimensional). □

In fact for most geometric purposes we need only the exterior algebra up to degree 2, so we limit ourselves to the general result about dimensions. In principle one may go on to compute explicit relations in higher degree and a Hodge $*$ operator as in [5] using the metric below, etc. The result is an important reminder that the degree 2 relations alone may not be enough for a geometrically reasonable exterior algebra.

Proposition 3.4. *In the setting of* Proposition 3.2 *above and for the framing defined by the Maurer–Cartan one-form, the moduli space of torsion-free connections is* $3|G|$*-dimensional and is given by the following component one-forms*:

$$A_t = (1+\alpha)e_t + \gamma e_x + \lambda e_y + \beta e_z, \qquad A_x = \lambda e_t + (1+\beta)e_x + \alpha e_y + \gamma e_z,$$
$$A_y = \beta e_t + \lambda e_x + (1+\gamma)e_y + \alpha e_z, \qquad A_z = \gamma e_t + \alpha e_x + \beta e_y + (1+\lambda)e_z, \quad (20)$$

where $\alpha, \beta, \gamma, \lambda$ *are functions on G such that*

$$\alpha + \beta + \gamma + \lambda = -1. \quad (21)$$

Thus, we have also

$$\sum_{a\in\mathcal{C}} A_a = 0. \quad (22)$$

Proof. We follow the same method as for S_3 in [2]. In the framing defined by the Maurer–Cartan one-form, the torsion-free connections obey the following equation (see Eq. (8)):

$$\sum_{b\in\mathcal{C}} A_b \wedge (e_{b^{-1}ab} - e_a) + \sum_{b\in\mathcal{C}} (e_b \wedge e_a + e_a \wedge e_b) = 0, \quad \forall a \in \mathcal{C}. \quad (23)$$

Using Table 1, we write (23) as

$$\begin{aligned} &A_x \wedge (e_z - e_t) + A_y \wedge (e_x - e_t) + A_z \wedge (e_y - e_t) \\ &\quad + (e_x + e_y + e_z) \wedge e_t + e_t \wedge (e_x + e_y + e_z) = 0, \\ &A_t \wedge (e_y - e_x) + A_y \wedge (e_z - e_x) + A_z \wedge (e_t - e_x) \\ &\quad + (e_t + e_y + e_z) \wedge e_x + e_x \wedge (e_t + e_y + e_z) = 0, \\ &A_t \wedge (e_z - e_y) + A_x \wedge (e_t - e_y) + A_z \wedge (e_x - e_y) \\ &\quad + (e_t + e_x + e_z) \wedge e_y + e_y \wedge (e_t + e_x + e_z) = 0, \\ &A_t \wedge (e_x - e_z) + A_x \wedge (e_y - e_z) + A_y \wedge (e_t - e_z) \\ &\quad + (e_t + e_x + e_y) \wedge e_z + e_z \wedge (e_t + e_x + e_y) = 0. \end{aligned} \quad (24)$$

We just have to solve the first three equations since the fourth one in this system can be obtained from the other by simple summation. We set $A_a = A_a^b e_b$ (sum over $b \in \mathcal{C}$) for functions $A_a^b \in H$ with

$$A_t^t = 1 + \alpha, \qquad A_x^x = 1 + \beta, \qquad A_y^y = 1 + \gamma, \qquad A_z^z = 1 + \lambda. \tag{25}$$

We put this into the equations to be solved and write them in the basis (19). Using the fact that each coefficient of the basis element has to vanish, we obtain

$$\begin{aligned} &A_x^t = \lambda = A_y^x, \qquad A_y^t = \beta = A_z^y = A_t^z, \qquad A_z^t = \gamma = A_t^x = A_x^z,\\ &A_z^x = -1 - \lambda - \gamma - \beta = A_x^y = A_y^z, \qquad A_t^y = -1 - \alpha - \beta - \gamma,\\ &\alpha + \beta + \gamma + \lambda = -1 \end{aligned} \tag{26}$$

as stated. Finally using these solutions one checks by simple computation that $A_t + A_x + A_y + A_z = 0$. □

We now study the regularity of connections.

Proposition 3.5. *Under the hypothesis of* Proposition 3.2 *the regular connections are either solutions of the system*:

$$\begin{aligned} &A_t \wedge A_t + A_x \wedge A_y + A_y \wedge A_z + A_z \wedge A_x = 0,\\ &A_t \wedge A_x + A_x \wedge A_z + A_y \wedge A_y + A_z \wedge A_t = 0,\\ &A_t \wedge A_y + A_x \wedge A_t + A_y \wedge A_x + A_z \wedge A_z = 0,\\ &A_t \wedge A_z + A_x \wedge A_x + A_y \wedge A_t + A_z \wedge A_y = 0, \end{aligned} \tag{27}$$

or solutions of the system:

$$\begin{aligned} &A_t \wedge A_t = 0, \qquad A_x \wedge A_x = 0, \qquad A_y \wedge A_y = 0, \qquad A_z \wedge A_z = 0,\\ &A_t \wedge A_x + A_x \wedge A_z + A_z \wedge A_t = 0, \qquad A_t \wedge A_y + A_x \wedge A_t + A_y \wedge A_x = 0,\\ &A_t \wedge A_z + A_y \wedge A_t + A_z \wedge A_y = 0, \qquad A_x \wedge A_y + A_y \wedge A_z + A_z \wedge A_x = 0. \end{aligned} \tag{28}$$

Proof. The general form of the regularity's equation is given by (9). One then needs the multiplication table at least for the elements of the class $\mathcal{C}$, by enumeration of the cases we find that under the hypothesis of Proposition 3.2, the only possible cases are those shown in Tables 2 and 3 . These correspond to the two possibilities stated. □

Table 2

×	t	x	y	z
t	t^2	zt	xt	yt
x	xt	x^2	xy	zt
y	yt	xt	y^2	xy
z	zt	xy	yt	z^2

Table 3

$\times$	t	x	y	z
t	t^2	zt	xt	yt
x	xt	yt	t^2	zt
y	yt	xt	zt	t^2
z	zt	t^2	yt	xt

The case of Table 2 corresponds for instance to the group $\mathrm{SL}(2, \mathbb{Z}_3)$ of the 2×2 matrices with coefficients in $\mathbb{Z}_3$, with any of its four-elements conjugacy classes (in the table any square is different from the products in (17)), while the case of Table 3 corresponds for instance to the alternating group $\mathcal{A}_4$ of order 12, with any of its four-elements conjugacy classes. To explicitly solve these nonlinear systems (27) and (28) one needs more precision on the group G. We solve system (27) in detail in Section 4 for $\mathcal{A}_4$. There is in fact a fundamental difference between the two cases, for instance the connection corresponding in (20) to $\alpha = \beta = \gamma = \lambda$ is a solution of (27) but not a solution of (28).

We also want to find the 'Levi-Civita connection', namely a regular torsion-free and cotorsion-free connection for a natural metric. We need for that end to find a suitable coframing or metric. As shown in [2] a natural choice in the group or quantum group case is to take any Ad-invariant nondegenerate bilinear form η defined on Ω_0^*, and indeed [2] provides a general 'braided-Killing form' construction that can achieve this. Our 'cyclic' conjugacy class $\mathcal{C}$ described above is not, however, semi-simple in the sense of Proposition 5.4 of [2] (the braided-Killing form is degenerate) and we instead have to determine all possible η.

Theorem 3.6. *Let $\mathcal{C}$ be a cyclic conjugacy class with n elements. Then up to normalisation, all nondegenerate Ad-invariant bilinear forms on Ω_0^* are given by*

$$\eta^{a,b} = \delta_{a,b} + \mu \tag{29}$$

for a constant $\mu \neq -1/n$. The associated metric in the Maurer–Cartan framing is

$$g = \sum_{a \in \mathcal{C}} e_a \underset{H}{\otimes} e_a + \mu\theta \underset{H}{\otimes} \theta.$$

Proof. Here g corresponds to an element $\eta \in \Omega_0 \otimes \Omega_0$ with coefficients $\eta^{a,b}$. We require it to be Ad-invariant and for the matrix of coefficients to be invertible (this is said more abstractly in [2] to handle the quantum group case). The first condition is easily seen to be the requirement

$$\eta^{g^{-1}ag,b} = \eta^{a,gbg^{-1}}, \quad \forall a, b \in \mathcal{C}, \quad g \in G. \tag{30}$$

This and nondegeneracy is easy to see for the η as stated.

Conversely, let us suppose that η is Ad-invariant and show that it is of the form (29). Since η is Ad-invariant, it obeys (30). We assume the existence of $t \in \mathcal{C}$ as in

Definition 3.1, then Ad_t is a cyclic permutation of $\mathcal{C}-\{t\}$. From invariance (30) it is obvious that $\eta^{t,a}=\eta^{t,\mathrm{Ad}_t(a)}=\eta^{t,\mathrm{Ad}_{t^2}(a)}=\cdots=\eta^{t,\mathrm{Ad}_{t^{n-2}}(a)}$ for any $a\neq t$, and hence by cyclicity

$$\eta^{t,b}=\mu_1,\quad \forall b\neq t$$

for some constant μ_1. But also from cyclicity we know that for any $a\in\mathcal{C}$ there is an element $c\in\mathcal{C}$ such that $a=ctc^{-1}$. Hence, from (30) we also have

$$\eta^{a,b}=\eta^{ctc^{-1},b}=\eta^{t,c^{-1}bc}=\mu_1,\quad \forall a\neq b, \tag{31}$$

so all off-diagonals are μ_1. Similarly, we have $\eta^{a,a}=\eta^{ctc^{-1},ctc^{-1}}=\eta^{t,t}=\mu_2$ for all $a\in\mathcal{C}$ by Ad-invariance, for some constant μ_2. Thus, $\eta^{a,b}=(\mu_2-\mu_1)\delta_{a,b}+\mu_1$, which has, up to an overall scaling, the form stated. The remaining condition on the parameter μ comes from the fact that η is invertible. Finally, given η we define

$$e^{*a}=\sum_{b\in\mathcal{C}}e_b\eta^{ba}$$

as explained in [2] for the associated coframing, which corresponds to the metric g as stated. □

One can then observe that this metric is symmetric in the sense

$$\wedge g=0. \tag{32}$$

The groups S_3, $\mathrm{SL}(2,\mathbb{Z}_3)$ and $\mathcal{A}_4$ are the examples of groups which obey the hypothesis of the previous theorem. The theorem clarifies the observation in [2] for S_3 where $\eta^{a,b}=\delta^{a,b}$ is derived as the braided-Killing form (up to a normalisation) but it is explained that one may add a multiple $\mu\theta\otimes_H\theta$ to the metric (without changing the connection and Riemannian curvature). We are now ready to describe torsion-free and cotorsion-free connections in our cyclic case.

Proposition 3.7. *In the setting of* Propositions 3.2 and 3.4 *and for the coframing given by* η *as above, the torsion-free and cotorsion-free connections obey the following relations*:

$$\begin{aligned}
&R_t^{-1}(\alpha)=R_x^{-1}(\lambda)=R_y^{-1}(\beta)=R_z^{-1}(\gamma),\\
&R_t^{-1}(\lambda)=R_x^{-1}(\alpha)=R_y^{-1}(\gamma)=R_z^{-1}(\beta),\\
&R_t^{-1}(\beta)=R_x^{-1}(\gamma)=R_y^{-1}(\alpha)=R_z^{-1}(\lambda),\\
&R_t^{-1}(\gamma)=R_x^{-1}(\beta)=R_y^{-1}(\lambda)=R_z^{-1}(\alpha),
\end{aligned} \tag{33}$$

where $\alpha,\beta,\gamma,\lambda$ *are as in* Proposition 3.4.

Proof. As in [2], when the coframing is given by the framing and an Ad-invariant η, one may easily compute the form of the cotorsion. One has,

$$D_Ae^{*a}=\eta^{ba}\,\mathrm{d}e_b+\sum_{b\in\mathcal{C},c\in\mathcal{C}}\eta^{ba}e_{cbc^{-1}}\wedge A_c-\sum_{b,c\in\mathcal{C}}\eta^{ba}e_b\wedge A_c$$

as a special case of the quantum groups computation in [2]. Since we suppose the connections to be torsion-free, Eq. (22) holds, then (cancelling η^{ba}), vanishing of cotorsion in Eq. (14) can be written equivalently as

$$\mathrm{d}e_a + \sum_{b\in\mathcal{C}} e_{bab^{-1}} \wedge A_b = 0, \quad \forall a \in \mathcal{C}. \tag{34}$$

If we write Eq. (34) for $a = t, x, y, z$, respectively, using Table 1, Eq. (18) and the definition of η, we obtain the following system of equations:

$$\begin{aligned}
&e_t \wedge A_t + e_x \wedge A_z + e_y \wedge A_x + e_z \wedge A_y - e_x \wedge e_z - e_y \wedge e_x - e_z \wedge e_y = 0,\\
&e_t \wedge A_z + e_x \wedge A_t + e_y \wedge A_y + e_z \wedge A_x - e_z \wedge e_x - e_x \wedge e_t - e_t \wedge e_z = 0,\\
&e_t \wedge A_x + e_x \wedge A_y + e_y \wedge A_t + e_z \wedge A_z - e_t \wedge e_x - e_x \wedge e_y - e_y \wedge e_t = 0,\\
&e_t \wedge A_y + e_x \wedge A_x + e_y \wedge A_z + e_z \wedge A_t - e_z \wedge e_t - e_t \wedge e_y - e_y \wedge e_z = 0.
\end{aligned} \tag{35}$$

We can get the fourth equation of system (35) from the three other. We then solve only the first three equations of this system. For that end, we set

$$A_a = e_b A'^b_a, \quad \forall a \in \mathcal{C} \tag{36}$$

with summation understood for $b \in \mathcal{C}$, and where we set

$$A'^t_t = 1 + \alpha', \qquad A'^x_x = 1 + \beta', \qquad A'^y_y = 1 + \gamma', \qquad A'^z_z = 1 + \lambda' \tag{37}$$

as above. We then proceed in the same manner as we solved system (24), using this time the right module structure of $\Omega^1(H)$. We find that the solutions (A_a) take the form

$$\begin{aligned}
&A_t = e_t(1+\alpha') + e_x\lambda' + e_y\beta' + e_z\gamma', \qquad A_x = e_t\gamma' + e_x(1+\beta') + e_y\lambda' + e_z\alpha',\\
&A_y = e_t\lambda' + e_x\alpha' + e_y(1+\gamma') + e_z\beta', \qquad A_z = e_t\beta' + e_x\gamma' + e_y\alpha' + e_z(1+\lambda')\\
&\text{with } \alpha' + \beta' + \gamma' + \lambda' = -1, \ \ \alpha', \beta', \gamma', \lambda' \in H.
\end{aligned} \tag{38}$$

We then write these solutions using the left module structure on one-forms via the commutation relation in (2). And we compare the result to that of system (20) to obtain system (33) as stated. □

At this level, we get many torsion-free and cotorsion-free connections. As one can remark, these equations for the connection do not depend on the coefficient μ of $\theta \otimes_H \theta$, just as was the case for S_3 in [2]. Modulo these modes, we see that there is an essentially unique form of invariant metric on G and we have given some conditions for the associated moduli of torsion-free and cotorsion-free regular connections.

4. Riemannian geometry of $\mathcal{A}_4$

In this section we specialise to the group $\mathcal{A}_4$ and present stronger results that depend on its structure and not only on the cyclic form of the conjugacy class. The group is defined by

$$\mathcal{A}_4 = \{e, u, v, w, t, x, y, z, t^2, ut^2, vt^2, wt^2\}, \tag{39}$$

where e is the group identity (this should not be confused with the Maurer–Cartan one-form) and t, u, v, w are the following permutations of $\{1, 2, 3, 4\}$:

$$t = (123), \qquad u = (14)(23), \qquad v = (12)(34), \qquad w = (13)(24), \tag{40}$$

and

$$x = tv = ut = (134), \qquad y = tw = vt = (243), \qquad z = tu = wt = (142). \tag{41}$$

The other products are

$$\begin{aligned} &v^2 = e, \quad w^2 = e, \quad u^2 = e, \quad t^3 = e, \quad vw = wv = u, \\ &vu = uv = w, \quad wu = uw = v, \end{aligned} \tag{42}$$

and we choose the conjugacy class

$$\mathcal{C} = \{t, x, y, z\}, \tag{43}$$

which is 'cyclic'. Indeed, we have

$$\begin{aligned} &\mathrm{Ad}_t(x) = txt^2 = t(ut)t^2 = z, \qquad \mathrm{Ad}_t(y) = tyt^2 = t(vt)t^2 = x, \\ &\mathrm{Ad}_t(z) = tzt^2 = t(wt)t^2 = y, \end{aligned}$$

and

$$\mathrm{Ad}_x(t) = xtyt = y, \qquad \mathrm{Ad}_y(t) = ytzt = z, \qquad \mathrm{Ad}_z(t) = ztxt = x, \qquad \mathrm{Ad}_t(t) = t,$$

which show that $\mathcal{C}$ obeys the conditions of Definition 3.1.

One may also check that the multiplication table of $\mathcal{C}$ corresponds to Table 3 as announced, so that we have at least one regular torsion-free and cotorsion-free connection on the bundle $H \otimes H$, where H denotes from now $\mathbb{C}[\mathcal{A}_4]$.

Proposition 4.1. *For the cyclic conjugacy class on $\mathcal{A}_4$, framing defined by the Maurer–Cartan form and coframing e^* by Ad-invariant η, there exists a unique 'Levi-Civita' connection, with component one-forms*

$$A_a = e_a - \tfrac{1}{4}\theta, \quad \forall a \in \mathcal{C}. \tag{44}$$

Proof. The connection defined in (44) is easily seen to be a solution of systems (20) and (38). We are going to show that it is the unique torsion-free and cotorsion-free connection which is solution of system (27). Using the properties of operators $(R_g)_{g\in\mathcal{A}_4}$ and equations of system (33) we find that

$$\alpha = R_u(\lambda), \qquad \beta = R_w(\lambda), \qquad \gamma = R_v(\lambda), \tag{45}$$

where λ is any function on $\mathcal{A}_4$ which obeys

$$(R_u + R_v + R_w + \mathrm{id})(\lambda) = -1. \tag{46}$$

At this level, λ is not necessarily a scalar. To determine it, we set

$$\lambda = \sum_{g\in\mathcal{A}_4} \lambda_g \delta_g, \tag{47}$$

hence we get from (46) that

$$\lambda = (-1-\lambda_v-\lambda_w-\lambda_u)\delta_e+\lambda_v\delta_v+\lambda_w\delta_w+\lambda_u\delta_u+(-1-\lambda_x-\lambda_y-\lambda_z)\delta_t \\ +\lambda_x\delta_x+\lambda_y\delta_y+\lambda_z\delta_z+(-1-\lambda_{tx}-\lambda_{ty}-\lambda_{tz})\delta_{t^2}+\lambda_{tx}\delta_{tx}+\lambda_{ty}\delta_{ty}+\lambda_{tz}\delta_{tz}.$$

We write out the first equation of system (27) in the basis (19) of $\Omega^2(H)$, passing from the right module structure to the left one, then set to zero each coefficient of the basis element and obtain the following equations:

$$\begin{aligned}
&(1+\alpha)R_1(\gamma)+\lambda R_1(\lambda)+\beta R_1(\alpha)+\gamma R_1(1+\beta)-\beta R_4(1+\alpha)-\gamma R_4(\beta)\\
&\quad-\alpha R_4(\gamma)-(1+\lambda)R_4(\lambda)=0,\\
&(1+\alpha)R_1(\lambda)+\lambda R_1(1+\gamma)+\beta R_1(\beta)+\gamma R_1(\alpha)-\lambda R_3(\gamma)-\alpha R_3(\lambda)\\
&\quad-(1+\gamma)R_3(\alpha)-\beta R_3(1+\beta)=0,\\
&(1+\alpha)R_1(\beta)+\lambda R_1(\alpha)+\beta R_1(1+\lambda)+\gamma R_1(\gamma)-\beta R_4(\lambda)-\gamma R_4(1+\gamma)\\
&\quad-\alpha R_4(\beta)-(1+\lambda)R_4(\alpha)=0,\\
&\gamma R_2(1+\alpha)+(1+\beta)R_2(\beta)+\lambda R_2(\gamma)+\alpha R_2(\lambda)-\lambda R_3(\gamma)-\alpha R_3(\lambda)\\
&\quad-(1+\gamma)R_3(\alpha)-\beta R_3(1+\beta)=0,\\
&\gamma R_2(\lambda)+(1+\beta)R_2(1+\gamma)+\lambda R_2(\beta)+\alpha R_2(\alpha)-\beta R_4(\gamma)-\gamma R_4(\lambda)\\
&\quad-\alpha R_4(\alpha)-(1+\lambda)R_4(1+\beta)=0,\\
&\gamma R_2(\beta)+(1+\beta)R_2(\alpha)+\lambda R_2(1+\lambda)+\alpha R_2(\gamma)-\beta R_4(1+\alpha)-\gamma R_4(\beta)\\
&\quad-\alpha R_4(\gamma)-(1+\lambda)R_4(\lambda)=0,\\
&\lambda R_3(1+\alpha)+\alpha R_3(\beta)+(1+\gamma)R_3(\gamma)+\beta R_3(\lambda)-\beta R_4(\lambda)-\gamma R_4(1+\gamma)\\
&\quad-\alpha R_4(\beta)-(1+\lambda)R_4(\alpha)=0,\\
&\lambda R_3(\beta)+\alpha R_3(\alpha)+(1+\gamma)R_3(1+\lambda)+\beta R_3(\gamma)-\beta R_4(\gamma)-\gamma R_4(\lambda)\\
&\quad-\alpha R_4(\alpha)-(1+\lambda)R_4(1+\beta)=0,
\end{aligned}\tag{48}$$

where the indexes 1–4 refer, respectively, to t, x, y, z. We then use (45) to write these equations, respectively, in terms of λ, then in terms of its scalar components. A long but straightforward computation of these components leads to $\lambda_g = -1/4, \forall g \in \mathcal{A}_4$, hence as an element of H, $\lambda = -1/4$. From (45), we also have $\alpha = \beta = \gamma = -1/4$. The expression of the corresponding connection in (20) is then as stated. To end the proof of Proposition 4.1, one checks easily that this connection is also a solution of the other equations of system (27). □

We refer to this connection as *the* 'Levi-Civita connection' for the invariant metric on the group $\mathcal{A}_4$.

Proposition 4.2. *The covariant derivative* $\nabla : \Omega^1(H) \to \Omega^1(H)\otimes_H\Omega^1(H)$ *for the 'Levi-Civita connection' on* $\mathcal{A}_4$, *and its Riemann curvature* $\mathcal{R} : \Omega^1(H) \to \Omega^2(H)\otimes_H$

$\Omega^1(H)$ are given by

$$\begin{aligned}
\nabla(e_t) &= -e_t \underset{H}{\otimes} e_t - e_x \underset{H}{\otimes} e_z - e_y \underset{H}{\otimes} e_x - e_z \underset{H}{\otimes} e_y + \tfrac{1}{4}\theta \underset{H}{\otimes} \theta,\\
\nabla(e_x) &= -e_t \underset{H}{\otimes} e_y - e_x \underset{H}{\otimes} e_x - e_y \underset{H}{\otimes} e_z - e_z \underset{H}{\otimes} e_t + \tfrac{1}{4}\theta \underset{H}{\otimes} \theta,\\
\nabla(e_y) &= -e_t \underset{H}{\otimes} e_z - e_x \underset{H}{\otimes} e_t - e_y \underset{H}{\otimes} e_y - e_z \underset{H}{\otimes} e_x + \tfrac{1}{4}\theta \underset{H}{\otimes} \theta,\\
\nabla(e_z) &= -e_t \underset{H}{\otimes} e_x - e_x \underset{H}{\otimes} e_y - e_y \underset{H}{\otimes} e_t - e_z \underset{H}{\otimes} e_z + \tfrac{1}{4}\theta \underset{H}{\otimes} \theta,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\mathcal{R}(e_t) &= \mathrm{d}e_t \underset{H}{\otimes} e_t + \mathrm{d}e_x \underset{H}{\otimes} e_z + \mathrm{d}e_y \underset{H}{\otimes} e_x + \mathrm{d}e_z \underset{H}{\otimes} e_y,\\
\mathcal{R}(e_x) &= \mathrm{d}e_t \underset{H}{\otimes} e_y + \mathrm{d}e_x \underset{H}{\otimes} e_x + \mathrm{d}e_y \underset{H}{\otimes} e_z + \mathrm{d}e_z \underset{H}{\otimes} e_t,\\
\mathcal{R}(e_y) &= \mathrm{d}e_t \underset{H}{\otimes} e_z + \mathrm{d}e_x \underset{H}{\otimes} e_t + \mathrm{d}e_y \underset{H}{\otimes} e_y + \mathrm{d}e_z \underset{H}{\otimes} e_x,\\
\mathcal{R}(e_z) &= \mathrm{d}e_t \underset{H}{\otimes} e_x + \mathrm{d}e_x \underset{H}{\otimes} e_y + \mathrm{d}e_y \underset{H}{\otimes} e_t + \mathrm{d}e_z \underset{H}{\otimes} e_z.
\end{aligned} \tag{50}$$

Proof. The curvature two-form F is defined by Eq. (10). In the present case, we have $bc \notin \mathcal{C}, \forall b, c \in \mathcal{C}$, so that $\sum_{b,c\in\mathcal{C},bc=a} A_b \wedge A_c = 0, \forall a \in \mathcal{C}$. We have also $\sum_{a\in\mathcal{C}} A_a = 0$ and $\mathrm{d}\theta = 0$, hence $F_a = \mathrm{d}A_a = \mathrm{d}e_a$ for the form of the connection in (44). This is exactly the same argument as for S_3 in [2]. Next, if we replace α in formula (11) by e_t, e_x, e_y, e_z, respectively, and use Table 1, we obtain relations (50) for the curvature. Finally, we compute the value of the covariant derivative on the basis one-forms $\{e_a\}$ by using formula (7). Explicitly, we have

$$\begin{aligned}
\nabla(e_a) &= -\sum_{b\in\mathcal{C}} A_b \underset{H}{\otimes} (e_{b^{-1}ab} - e_a) = -\sum_{b\in\mathcal{C}} \left(e_b - \frac{1}{4}\theta\right) \underset{H}{\otimes} (e_{b^{-1}ab} - e_a)\\
&= -\sum_{b\in\mathcal{C}} e_b \underset{H}{\otimes} (e_{b^{-1}ab} - e_a) + \frac{1}{4}\theta \underset{H}{\otimes} \sum_{b\in\mathcal{C}} (e_{b^{-1}ab} - e_a)\\
&= -\sum_{b\in\mathcal{C}} e_b \underset{H}{\otimes} e_{b^{-1}ab} + \sum_{b\in\mathcal{C}} e_b \underset{H}{\otimes} e_a + \frac{1}{4}\theta \underset{H}{\otimes} \sum_{b\in\mathcal{C}} (e_b - e_a)\\
&= -\sum_{b\in\mathcal{C}} e_b \underset{H}{\otimes} e_{b^{-1}ab} + \frac{1}{4}\theta \underset{H}{\otimes} \theta.
\end{aligned}$$

According to Table 1, this last equation gives relations (49) as stated. □

From the Riemann curvature and the canonical lift i we can compute the Ricci curvature of the Levi-Civita connection on $\mathcal{A}_4$ and find that it vanishes. In fact we can prove a slightly stronger result that is it the *only* Ricci flat connection for this choice of framing.

Theorem 4.3. *For the framing defined by the Maurer–Cartan one-form, and for the canonical lift i, the above Levi–Civita connection on $\mathcal{A}_4$ is the unique regular Ricci flat connection.*

Proof. In the present case, the canonical lift takes the form

$$i(e_a \wedge e_b) = e_a \underset{H}{\otimes} e_b - \frac{1}{3} \sum_{cd=ab, c\neq d} e_c \underset{H}{\otimes} e_d, \qquad i(e_a \wedge e_a) = 0. \tag{51}$$

We have to solve for vanishing of [2]

$$\mathrm{Ricci} = \sum_{a\in\mathcal{C}} \langle f^a, (i \underset{H}{\otimes} \mathrm{id})\mathcal{R}(e_a)\rangle = \sum_{a,b,c\in\mathcal{C}} i(F_c)^{ab} e_b \underset{H}{\otimes} (e_{c^{-1}ac} - e_a),$$

where $i(F_c) = i(F_c)^{ab} e_a \otimes_H e_b$, and the pairing is made between each f^a and the first factor of the tensor product $(i\otimes_H \mathrm{id})\mathcal{R}(e_a)$ according to the formula $\langle f^a, me_b\rangle = m\delta^a_b, \forall m \in H$. In our case this becomes

$$\begin{aligned}
&\langle f^t, i(F_x)\underset{H}{\otimes}(e_z - e_t) + i(F_y)\underset{H}{\otimes}(e_x - e_t) + i(F_z)\underset{H}{\otimes}(e_y - e_t)\rangle \\
&\quad +\langle f^x, i(F_t)\underset{H}{\otimes}(e_y - e_x) + i(F_y)\underset{H}{\otimes}(e_z - e_x) + i(F_z)\underset{H}{\otimes}(e_t - e_x)\rangle \\
&\quad +\langle f^y, i(F_t)\underset{H}{\otimes}(e_z - e_y) + i(F_x)\underset{H}{\otimes}(e_t - e_y) + i(F_z)\underset{H}{\otimes}(e_x - e_y)\rangle \\
&\quad +\langle f^z, i(F_t)\underset{H}{\otimes}(e_x - e_z) + i(F_x)\underset{H}{\otimes}(e_y - e_z) + i(F_y)\underset{H}{\otimes}(e_t - e_z)\rangle = 0.
\end{aligned} \tag{52}$$

We first compute F_t, F_x, F_y, F_z and $i(F_t), i(F_x), i(F_y), i(F_z)$ for general free torsion connections given in Proposition 3.4, then we rewrite Eq. (52) in terms of the basic elements $\{e_a\otimes_H e_b\}_{a,b\in\mathcal{C}}$ of the left H-module $\Omega^1(H)\otimes_H\Omega^1(H)$, the vanishing of each coefficient of the mentioned basic elements leads to 16 equations in terms of $\alpha, \beta, \gamma, \lambda$ and their 'first-order derivatives' $\partial^a\alpha, \partial^a\beta, \partial^a\gamma, \partial^a\lambda, a \in \mathcal{C}$.

We find that it is enough to solve the following four equations coming from the coefficients of $e_t\otimes_H e_t$, $e_x\otimes_H e_x$, $e_y\otimes_H e_y$, $e_z\otimes_H e_z$, respectively,

$$\begin{aligned}
&\beta - 2\gamma + \lambda + \partial^y\alpha + \partial^z\alpha + \partial^x\alpha + \partial^t\beta - 2\partial^z\beta + \partial^t\gamma - 2\partial^x\gamma + \partial^t\lambda - 2\partial^y\lambda = 0, \\
&\alpha - 3\beta + \lambda + \gamma + \partial^x\alpha - 2\partial^y\alpha + \partial^y\beta + \partial^t\beta + \partial^z\beta + \partial^t\gamma - 2\partial^z\gamma + \partial^x\lambda - 2\partial^t\lambda = 0, \\
&\alpha + \beta - 3\gamma + \lambda + \partial^y\alpha - 2\partial^z\alpha + \partial^y\beta - 2\partial^t\beta + \partial^x\gamma + \partial^t\gamma + \partial^z\gamma + \partial^y\lambda - 2\partial^x\lambda = 0, \\
&\alpha + \beta + \gamma - 3\lambda + \partial^z\alpha - 2\partial^x\alpha + \partial^z\beta - 2\partial^y\beta + \partial^z\gamma - 2\partial^t\gamma + \partial^y\lambda + \partial^t\lambda + \partial^x\lambda = 0.
\end{aligned} \tag{53}$$

Indeed, we transform these four equations to a system of 48 linear equations where the variables are the components of α, β, γ and λ in the basis $(\delta_g)_{g\in\mathcal{A}_4}$. The unique solution of the mentioned system which obeys the condition $\alpha + \beta + \gamma + \lambda = -1$ as in Proposition 3.4 is $\alpha = \beta = \gamma = \lambda = -1/4$. To end the proof one just checks easily that this solution is also a solution of the 12 remaining equations (of the 16 mentioned above), coming from the coefficients of $e_a\otimes_H e_b$, $a \neq b$ in Eq. (52). □

One can also check that the Ricci tensor for the Levi-Civita connection with respect to the alternative 'lift'

$$i'(e_a \wedge e_b) = e_a \underset{H}{\otimes} e_b - e_{aba^{-1}} \underset{H}{\otimes} e_a \tag{54}$$

also vanishes, i.e. the result does not depend strongly on the choice of lift. This is the same as found for S_3, where the two Ricci tensors with respect to i and i', respectively, are the same up to a scale [2].

5. The Dirac operator for $\mathcal{A}_4$

Following the formalism of [2], we write down in this section the 'gamma-matrices' and the Dirac operator associated to the Maurer–Cartan framing e and the coframing e^* for the invariant metric. We use the associated Levi-Civita connection constructed above.

For the 'spinor' representation, we consider the standard three-dimensional representation of $\mathcal{A}_4$ defined on a vector space W by

$$\rho_W(t)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\qquad \rho_W(u)=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},$$
$$\rho_W(v)=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\qquad \rho_W(w)=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \tag{55}$$

and e is the unit matrix I. The Casimir element C associated to the operator η is given in [2] by

$$C=\eta_{ab}^{-1}f^af^b=\eta_{ab}^{-1}(a-e)(b-e) \tag{56}$$

with summation understood, $b, a\in\mathcal{C}$. One checks that it corresponds in the general case of the class $\mathcal{C}=\{t,x,y,z\}$ as in Proposition 3.2, to the explicit form

$$\begin{aligned}C={}&\frac{1+3\mu}{1+4\mu}[t^2+x^2+y^2+z^2-2(t+x+y+z)+4e]\\&+\frac{-3\mu}{1+4\mu}[tx+ty+tz+xy-2(t+x+y+z)+4e].\end{aligned} \tag{57}$$

In the case of $\mathcal{A}_4$, Eq. (57) reads

$$C=\frac{1}{1+4\mu}[(t-e)^2+(x-e)^2+(y-e)^2+(z-e)^2],$$

then

$$\rho_W(C)=\frac{4}{1+4\mu}I.$$

Next, we choose our gamma-matrix γ to be the 'tautological gamma-matrix' [2] associated to ρ_W and η defined by

$$\gamma_a=\eta_{ab}^{-1}\rho_W(f^b)=\sum_{b\in\mathcal{C}}\eta_{ab}^{-1}\rho_W(b-e),\qquad \forall a\in\mathcal{C}. \tag{58}$$

In our case we find

$$\gamma_a = \rho_W(a-e) + \frac{4\mu}{1+4\mu}, \tag{59}$$

and that these matrices obey the relations

$$\sum_{a\in\mathcal{C}} \gamma_a = -\frac{4}{1+4\mu}, \tag{60}$$

$$\gamma_a\gamma_b + \gamma_b\gamma_a + \frac{2}{1+4\mu}(\gamma_a+\gamma_b) + \frac{2}{(1+4\mu)^2} = \rho_W(ab+ba) \tag{61}$$

following directly from (59).

Eqs. (59)–(61) hold in the general case considered in Proposition 3.2, providing that the multiplication table is that of Table 3. The explicit matrix representation of these gamma-matrices above for $\mathcal{A}_4$ are as follows:

$$\gamma_t = \begin{pmatrix} \frac{-1}{1+4\mu} & 0 & 1 \\ 1 & \frac{-1}{1+4\mu} & 0 \\ 0 & 1 & \frac{-1}{1+4\mu} \end{pmatrix}, \quad \gamma_x = \begin{pmatrix} \frac{-1}{1+4\mu} & 0 & -1 \\ -1 & \frac{-1}{1+4\mu} & 0 \\ 0 & 1 & \frac{-1}{1+4\mu} \end{pmatrix},$$

$$\gamma_y = \begin{pmatrix} \frac{-1}{1+4\mu} & 0 & -1 \\ 1 & \frac{-1}{1+4\mu} & 0 \\ 0 & -1 & \frac{-1}{1+4\mu} \end{pmatrix}, \quad \gamma_z = \begin{pmatrix} \frac{-1}{1+4\mu} & 0 & 1 \\ -1 & \frac{-1}{1+4\mu} & 0 \\ 0 & -1 & \frac{-1}{1+4\mu} \end{pmatrix}. \tag{62}$$

Proposition 5.1. *The Dirac operator* (15) *on* $\mathcal{A}_4$ *for the gamma-matrices and the Levi-Civita connection on* $\mathcal{A}_4$ *constructed above is given by*

$$\not{D} = \partial^a\gamma_a - 4, \tag{63}$$

(*sum over* $a\in\mathcal{C}$). *For* $\mu=0$ *we have explicitly*

$$\not{D} = \begin{pmatrix} -R_t-R_x-R_y-R_z & 0 & R_t-R_x-R_y+R_z \\ R_t-R_x+R_y-R_z & -R_t-R_x-R_y-R_z & 0 \\ 0 & R_t+R_x-R_y-R_z & -R_t-R_x-R_y-R_z \end{pmatrix}.$$

This has 18 *zero modes*, 3 *modes with eigenvalue* ± 4, 3 *modes with eigenvalue* $\pm 4q$, *and* 3 *modes with eigenvalue* $\pm 4\bar{q}$, *where* $q = \mathrm{e}^{2\pi\imath/3}$.

Proof. The formula giving the Dirac operator in terms of the gamma-matrices and the representation ρ_W is given by Eq. (15). We first observe that for the representation ρ_W

above, the following two equations hold: $\sum_{a\in\mathcal{C}}\rho_W(a)=0$ and $\sum_{a\in\mathcal{C}}\rho_W(a^2)=0$. Using the A_a^b defined by (44), and the fact that every element of $\mathcal{C}$ is of order 3, we obtain

$$\begin{aligned}
\not{D} &= \partial^a\gamma_a - \sum_{a,b\in\mathcal{C}}\left(\delta_a^b-\frac{1}{4}\right)\left[\rho_W(a-e)+\frac{4\mu}{1+4\mu}\right]\rho_W(b^2-e)\\
&= \partial^a\gamma_a - \sum_{a\in\mathcal{C}}\left[\rho_W(a-e)+\frac{4\mu}{1+4\mu}\right]\rho_W(a^2-e)\\
&\quad + \sum_{a\in\mathcal{C}}\frac{1}{4}\left[\rho_W(a-e)+\frac{4\mu}{1+4\mu}\right](-4I)\\
&= \partial^a\gamma_a - \sum_{a\in\mathcal{C}}[\rho_W(a^3-a-a^2+e)] - \frac{4\mu}{1+4\mu}\sum_{a\in\mathcal{C}}\rho_W(a^2-e) - \sum_{a\in\mathcal{C}}\gamma_a = \partial^a\gamma_a - 4.
\end{aligned}$$

We then replace in Eq. (63) the representation of the gamma-matrices from (62) to obtain the matrix representation of $\not{D}$ as stated.

To compute its eigenvalues we need R_a explicitly as 12×12 matrices. In the basis spanned by delta functions at $\{e, u, v, w, t, x, y, z, t^2, ut^2, vt^2, wt^2\}$, the right translation operators take the form

$$R_t=\begin{pmatrix}0&I&0\\0&0&I\\I&0&0\end{pmatrix},\qquad R_x=\begin{pmatrix}0&Y&0\\0&0&Z\\X&0&0\end{pmatrix},$$
$$R_y=\begin{pmatrix}0&X&0\\0&0&Y\\Z&0&0\end{pmatrix},\qquad R_z=\begin{pmatrix}0&Z&0\\0&0&X\\Y&0&0\end{pmatrix},$$

where I is the 4×4 identity matrix and

$$X=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\qquad Y=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\qquad Z=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}.$$

We then obtain the eigenvalues as stated. □

The eigenvalues here do in fact depend on μ and the case $\mu=0$ seems to be the more natural since it corresponds to the simplest metric $\delta_{a,b}$. The -4 in (63) corresponds to the constant curvature of $\mathcal{A}_4$ as for S_3 in [2]. As for S_3, this offset ensures a symmetrical distribution of eigenvalues about zero.

We will now construct the eigenstates of $\not{D}$. Before doing that we look at the spin 0 or scalar wave equation defined by the corresponding wave operator

$$\Box = -\eta_{ab}^{-1}\partial^a\partial^b = -\sum_a\partial^a\partial^a = \sum_a(2R_a - R_{a^2} - \mathrm{id}). \tag{64}$$

We do not exactly expect a Lichnerowicz formula relating this to the square of $\not{D}$, but we find that $\Box$ is the square of a first-order operator with eigenvalues contained in those of $\not{D}$. It is easy to solve the wave equation directly.

Proposition 5.2.

$$\Box = -\frac{1}{4}\left(\sum_a \partial^a\right)^2 = -\frac{1}{4}(D_0 - 4)^2, \quad D_0 = \sum_a R_a.$$

There is one zero mode, given by the constant function. There is one mode of eigenvalue $12q$ and one of eigenvalue $12\bar{q}$ given by the two other one-dimensional representations of $\mathcal{A}_4$. Finally, there are nine modes with eigenvalue -4 given by the matrix elements of the remaining three-dimensional irreducible representation ρ_W above.

Proof. The square form of $\Box$ follows from the multiplication table shown in Table 3. From there one finds that $(\sum_a R_a)^2 = 4\sum_a R_{a^2}$, after which the result follows. To solve the wave equation, note that the nontrivial one-dimensional representations ρ, $\bar{\rho}$ of $\mathcal{A}_4$ are given by

$$\rho(t) = q, \quad \rho(u) = \rho(v) = \rho(w) = 1, \qquad \bar{\rho}(t) = \bar{q}, \quad \bar{\rho}(u) = \bar{\rho}(v) = \bar{\rho}(w) = 1.$$

Then $\forall m \in \mathcal{A}_4$,

$$\begin{aligned}\Box\rho(m) &= 2\sum_a \rho(m)\rho(a) - \sum_a \rho(m)\rho(a^2) - 4\rho(m)\\ &= (8q - 4 - 4\bar{q})\rho(m) = 12q\rho(m).\end{aligned}$$

Similarly for $\bar{\rho}$ with q replaced by $\bar{q}$. Finally for the matrix elements $\{\rho_{kl}\}$ of ρ_W, we have

$$\begin{aligned}\Box\rho_{kl}(m) &= \sum_a [2R_a\rho_{kl}(m) - R_{a^2}\rho_{kl}(m) - \rho_{kl}(m)]\\ &= \sum_a \sum_i [2\rho_{ki}(m)\rho_{il}(a) - \rho_{ki}(m)\rho_{il}(a^2)] - 4\rho_{kl}(m) = -4\rho_{kl}(m),\end{aligned}$$

since $\sum_a \rho_W(a^2) = 0$ and $\sum_a \rho_W(a) = 0$. The nine "waves" ρ_{kl} are linearly independent because the representation ρ_W is irreducible. Hence, we have completely diagonalised the 12×12 matrix $\Box$. Equivalently, we have diagonalised D_0 with corresponding eigenvalues $4, 4q, 4\bar{q}, 0$ as for $\not{D}$ above. $\Box$

Moreover, every function ϕ on $\mathcal{A}_4$ has a unique decomposition of the form

$$\phi = p_0 + p_1\rho + p_2\bar{\rho} + \sum_{k,l} p_{kl}\rho_{kl} \tag{65}$$

for some numbers p_0, p_1, p_2, p_{kl} which are the components of ϕ in the nonabelian Fourier transform. The decomposition above corresponds precisely to the Peter–Weyl decomposition, just as noted for S_3 in [5].

We now use the preceding results to completely solve the Dirac equation. We set

$$D_1 = R_t - R_x - R_y + R_z, \qquad D_2 = R_t - R_x + R_y - R_z,$$
$$D_3 = R_t + R_x - R_y - R_z,$$

so that

$$\not{D} = \begin{pmatrix} -D_0 & 0 & D_1 \\ D_2 & -D_0 & 0 \\ 0 & D_3 & -D_0 \end{pmatrix}.$$

Let us note first of all that

$$D_1 D_2 = 0, \qquad D_2 D_3 = 0, \qquad D_3 D_1 = 0,$$
$$D_0 D_i = 0, \qquad D_i^2 = 0, \quad 1 \le i \le 3 \tag{66}$$

from which we see by inspection that the following are 18 linearly-independent zero modes of $\not{D}$:

$$\begin{pmatrix} D_2\rho_{k3} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ D_3\rho_{k1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ D_1\rho_{k2} \end{pmatrix}, \qquad \begin{pmatrix} D_3\rho_{k1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ D_1\rho_{k2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ D_2\rho_{k3} \end{pmatrix} \tag{67}$$

for $1 \le k \le 3$. Similarly it is immediate by inspection that

$$\begin{pmatrix} \rho^n \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \rho^n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \rho^n \end{pmatrix} \tag{68}$$

are three modes with eigenvalue $-4q^n$, for $n = 0$, 1 and 2. This is because $D_0\rho = 4q\rho$ (as in Proposition 5.2 above) while $D_i\rho = 0$ for $i > 0$.

It remains only to construct the $+4q^n$ eigenmodes for $n = 0$, 1 and 2. Before doing this let us make two observations about the modes already evident. First of all, let $\hat{\rho}$ denote the operator of multiplication by ρ. Then $R_a\hat{\rho} = q\hat{\rho}R_a$ since $\rho(a) = q$ for $a = t, x, y, z$. Hence,

$$\not{D}\hat{\rho} = q\hat{\rho}\not{D}. \tag{69}$$

Thus, multiplication of a spinor mode by the function ρ multiplies its eigenvalue by q. This generates the $-4q^n$ modes above from the $n = 0$ case.

Secondly, from the multiplication table shown in Table 3 we see that

$$R_t D_1 = D_2 R_t, \tag{70}$$

and its cyclic rotations $1 \to 2 \to 3 \to 1$. From this one finds

$$[\chi, \not{D}] = 0, \qquad \chi^3 = \mathrm{id}, \quad \chi = \begin{pmatrix} 0 & 0 & R_t \\ R_t & 0 & 0 \\ 0 & R_t & 0 \end{pmatrix}. \tag{71}$$

This χ generates the other two modes from the first in each group of three in (67) and (68). In the case of the zero modes note that

$$R_t \rho_{k1} = \rho_{k2}, \tag{72}$$

and its cyclic rotations $1 \to 2 \to 3 \to 1$, from the explicit form of $\rho_W(t)$.

We now observe that if we make an ansatz of the form

$$\psi = \begin{pmatrix} \phi \\ R_t \phi \\ R_{t^2}\phi \end{pmatrix} = \begin{pmatrix} \mathrm{id} \\ R_t \\ R_{t^2} \end{pmatrix} \phi$$

for function ϕ then

$$\not{D}\psi = \begin{pmatrix} \mathrm{id} \\ R_t \\ R_{t^2} \end{pmatrix} (-D_0 + R_{t^2} D_2)\phi.$$

So eigenspinors are induced by eigenfunctions of the operator

$$-D_0 + R_{t^2} D_2 = -D_0 + R_e - R_u - R_v + R_w.$$

All of the ρ_{kl} are zero modes of D_0 (as in Proposition 5.2), while among them precisely ρ_{k1} is an eigenmode of $R_e - R_u - R_v + R_w$, with eigenvalue 4 (this follows from $(1/4)(\rho_W(e) - \rho_W(u) - \rho_W(v) + \rho_W(w))$ being a projection matrix of rank 1). Hence, $\phi = \rho_{k1}$ in the ansatz yields three spinor modes

$$\begin{pmatrix} \rho_{k1} \\ \rho_{k2} \\ \rho_{k3} \end{pmatrix}, \quad 1 \le k \le 3 \tag{73}$$

with eigenvalue $+4$ of $\not{D}$. Applying $\hat{\rho}$ generates the three with eigenvalue $4q$ and then the three with eigenvalue $4\bar{q}$.

This completes our diagonalisation of $\not{D}$. Finally, we note that there necessarily exists an operator γ with $\gamma^2 = \mathrm{id}$ and $\{\gamma, \not{D}\} = 0$, but it is not unique. Thus, diagonalising $\not{D}$, we can group the eigenbasis into pairs of three-blocks of zero modes according to the two groups in (67), interchanged by γ, and similarly we define γ to interchange the two three-blocks with eigenvalues $\pm 4q^n$. This defines at least one choice of γ, suggested by our explicit diagonalisation.

6. Cohomology and concluding remarks

In this paper we have concentrated on the Riemannian geometry of $\mathcal{A}_4$. There are also some more elementary geometrical questions that one could look at, related to the differential structure alone. We will discuss some of them here.

Firstly, given an exterior algebra $\Omega(H)$ one has a noncommutative de Rahm cohomology defined as usual by closed forms modulo exact ones. We find, just as for S_3 in [5], that θ generates H^1.

Proposition 6.1. *For $\mathcal{A}_4$ with cyclic conjugacy class $\{t, x, y, z\}$ the first noncommutative de Rahm cohomology is*

$$H^1(\mathcal{A}_4) = \mathbb{C}\theta.$$

Proof. We compute $\partial^a = R_a - \mathrm{id}$ explicitly as four 12×12 matrices for their action on $\mathbb{C}[\mathcal{A}_4]$. The concatenation of these define $\mathrm{d}_0 : \mathbb{C}[\mathcal{A}_4] \to \mathbb{C}[\mathcal{A}_4] \otimes \Omega_0 = \Omega^1(H)$ as a 48×12 matrix. We also define the 8×16 matrix π which sends $e_a \otimes_H e_b \to e_a \wedge e_b$ using the tensor product 16-dimensional basis of $\Omega_0 \otimes \Omega_0$ and the eight-dimensional vector space over $\mathbb{C}$ with basis (19). Similarly, we define an 8×4 matrix for 'd' acting on Ω_0 again using the basis (19) over $\mathbb{C}$. From these ingredients, we build $\mathrm{d}_1 : \Omega^1(H) \to \Omega^2(H)$ as a 96×48 matrix defined by $\mathrm{d}_1(fe_a) = \mathrm{d}_0(f) \wedge e_a + f\, \mathrm{d}e_a$. We then compute the kernel of d_1 and find it to be 12-dimensional. The image of d_0 is necessarily 11-dimensional (its kernel is the constant functions) and hence the cohomology is one-dimensional. θ is closed but never exact (for any finite group) and hence represents this class. □

Next, whereas the cohomology is a linear problem, one can also consider its nonlinear variant called $U(1)$-gauge theory. Here we define the curvature of a one-form $\alpha \in \Omega^1(H)$ to be the two-form $F(\alpha) = \mathrm{d}\alpha + \alpha \wedge \alpha$. This transforms by conjugation under the gauge transform $\alpha \mapsto u\alpha u^{-1} + u\, \mathrm{d}u^{-1}$ for any nowhere zero function $u \in H$. One can also impose here unitarity conditions as in [5]. In this context it would be interesting to find the moduli space of (unitary) flat connections. This was done for S_3 in [5] and found to have a richer structure than the cohomology alone, i.e. with other solutions beyond multiples of θ and we would expect something similarly rich for $\mathcal{A}_4$. For example, if we focus on flat connections with constant coefficients in the $\{e_a\}$ basis as in [7], a short computation shows that these are given by the five lines

$$\lambda e_t - \theta, \qquad \lambda e_x - \theta, \qquad \lambda e_y - \theta, \qquad \lambda e_z - \theta, \qquad (\lambda - 1)\theta, \tag{74}$$

where λ is a parameter. There is a similar behaviour for any cyclic conjugacy class.

A further question relates to the fact that $\mathcal{A}_4 \subset S_4$ as a normal subgroup. Therefore, its exterior algebra should be related to that of S_4 for a suitable conjugacy class on that. The different differential structures on S_4, S_5 for different conjugacy classes are studied in [7] and looking there, one finds that the order 8 conjugacy class containing (123) in S_4 has the required exterior algebra. Its eight generators split into two sets, namely $\{e_{(123)}, e_{(134)}, e_{(243)}, e_{(142)}\}$ generating a subalgebra with relations as in Proposition 3.2 and a complementary set $\{e_{(132)}, e_{(143)}, e_{(234)}, e_{(124)}\}$ generating the opposite subalgebra. We denote the latter generators by $\{e_{\bar{t}}, e_{\bar{x}}, e_{\bar{y}}, e_{\bar{z}}\}$. There are nontrivial cross-relations between the two sets:

$$e_a \wedge e_{\bar{a}} + e_{\bar{a}} \wedge e_a = 0, \quad e_t \wedge e_{\bar{z}} + e_{\bar{z}} \wedge e_x + e_x \wedge e_{\bar{y}} + e_{\bar{y}} \wedge e_t = 0,$$
$$e_t \wedge e_{\bar{x}} + e_{\bar{x}} \wedge e_y + e_y \wedge e_{\bar{z}} + e_{\bar{z}} \wedge e_t = 0, \quad e_t \wedge e_{\bar{y}} plus e_{\bar{y}} \wedge e_z + e_z \wedge e_{\bar{x}} + e_{\bar{x}} \wedge e_t = 0, \tag{75}$$

and their three conjugates (given by applying an overbar and reversing products). In other words, the differential geometry of S_4 appears to be some form of 'complexification' of that of $\mathcal{A}_4$.

Indeed, the conjugacy class $\{\bar{t}, \bar{x}, \bar{y}, \bar{z}\}$ in $\mathcal{A}_4$ is also cyclic and the equations of its associated exterior algebra are of the same form as in Proposition 3.2. It defines a differential geometry on $\mathcal{A}_4$ conjugate to the one we have studied above. Indeed, one has

$$\partial_a^{\dagger} = \partial_{a^{-1}} = \partial_{a^2} = \partial_{\bar{a}}, \tag{76}$$

where dagger (†) denotes transpose with respect to the l^2 inner product on $\mathcal{A}_4$. Here the first equality is a general feature for any finite group and follows from the braided-Leibniz rule. The second equality is due to all elements of $\mathcal{C}$ in our case having order 3 and the third is a special feature $a^2 = \bar{a}$ of the multiplication table shown in Table 3. To complete the analysis let us note that $\mathcal{A}_4$ has just one other nontrivial conjugacy class $\{u, v, w\}$. This does not generate $\mathcal{A}_4$, i.e. the quantum manifold structure that it defines is not connected (not every point can be reached from any other by steps taken from the conjugacy class). The component of this connected to the group identity is $\mathbb{Z}_2 \times \mathbb{Z}_2$ with its universal differential calculus.

Finally, returning to the Riemannian geometry, one can and should consider more general metrics and vierbeins. Since we have found above a canonical invariant η, one could use this to fix the relationship between a general vierbein and covierbein, i.e. look for metrics of the form $g = \eta^{ab} e_a \otimes_H e_b$ with $\{e_a\}$ as the free variable (rather than e, e^* independent). Among the moduli space of pairs (e, A) of vierbein and spin connection (or more generally of triples (e, e^*, A)), our results above show that there is at least one canonical point where the Ricci tensor vanishes. This motivates the problem of solving the Ricci flat equations in general, i.e. classical 'gravity' on $\mathcal{A}_4$. Similarly, one can consider functional integrals, now a finite number of usual integrals, with Einstein–Hilbert action, i.e. 'quantum-gravity'. These are difficult nonlinear questions unsolved even for S_3 and beyond our present scope.

Acknowledgements

One of us (F.N.) would like to thank Prof. J.-P. Antoine for facilitating his stay at FYMA-UCL during 2000–2001. He also acknowledges support from CIUF-CUD. S.M. is a Royal Society University Research Fellow.

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